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Error Estimates for Interpolation by Compactly Supported Radial Basis Functions of Minimal Degree

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We consider error estimates for interpolation by a special class of compactly supported radial basis functions. These functions consist of a univariate polynomial within their support and are of minimal degree depending on space dimension and smoothness. Their associated "native" Hilbert spaces are shown to be norm-equivalent to Sobolev spaces. Thus we can derive approximation orders for functions from Sobolev spaces which are comparable to those of thin-plate-spline interpolation. Finally, we investigate the numerical stability of the interpolation process. © 1998 Academic Press

1. INTERPOLATION BY COMPACTLY SUPPORTED RADIAL BASIS FUNCTIONS

For a given function $f \in C(\mathbb{R}^d)$ the radial basis function interpolant s_f on a set $X = \{x_1, ..., x_N\} \subseteq \mathbb{R}^d$ of distinct points is given by

$$s_f(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j), \tag{1}$$

where $\Phi: \mathbb{R}^d \to \mathbb{R}$ is a fixed, usually radial function $\Phi(x) = \phi(\|x\|_2)$ and the coefficients $\alpha_1, ..., \alpha_N$ are determined by the interpolation conditions

$$s_f(x_j) = f(x_j), \qquad 1 \leqslant j \leqslant N. \tag{2}$$

A large number of centers x_j on the one hand or a large number of evaluations of the interpolating function (1) on the other hand makes it obviously desirable to have a compactly supported basis function Φ of the simplest possible form.

But the most popular Φ's are not compactly supported. They often do not even allow one to form the interpolant as a pure "radial" sum (1), so

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polynomials up to a certain degree are added. As we are interested in a special class of basis functions Φ which allow interpolants s_f of the form (1), we skip the details and restrict ourselves to positive definite functions, but refer the reader to the overview articles [2, 5, 11, 13, 14] for the more general setting.

DEFINITION 1.1. A continuous function $\Phi: \mathbb{R}^d \to \mathbb{R}$ is said to be positive definite iff for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{x_1, ..., x_N\} \subseteq \mathbb{R}^d$, and all vectors $\alpha \in \mathbb{R}^N \backslash \{0\}$ the quadratic form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k)$$

is positive. A univariate even function $\phi: \mathbb{R} \to \mathbb{R}$ is called positive definite on $\mathbb{R}^d$, which we abbreviate by $\phi \in \mathrm{PD}_d$, if the function $\Phi(x) = \phi(\|x\|_2)$, $x \in \mathbb{R}^d$, is positive definite.

This definition ensures that the interpolation problem (2) with s_f from (1) is uniquely solvable, because the interpolation matrices $A_{X,\Phi} = (\Phi(x_j - x_k))$ are positive definite.

Note that it suffices to determine the univariate function ϕ only for non-negative arguments because ϕ is even. Thus from now on we shall only consider univariate functions $\phi: \mathbb{R}_{\geqslant 0} \to \mathbb{R}$.

One advantage of the classical radial basis functions like Gaussians, thin-plate-splines, or multiquadratics is their simple representation, which holds for every space dimension d, i.e., the same univariate function ϕ can be used as a basis function $\Phi(x) = \phi(\|x\|_2)$ on every $\mathbb{R}^d$. As it is a simple consequence of a theorem of Schoenberg [15] that a compactly supported univariate function $\phi: \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ cannot be positive definite on every $\mathbb{R}^d$ we have to accept the dependence of ϕ on the space dimension d as soon as we work with a compact support. But this is actually no real disadvantage.

Now we introduce the operator I and its inverse D for $r \geqslant 0$ by

$$(I\phi)(r) = \int_r^\infty t\phi(t)\, dt,$$

$$(D\phi)(r) = -\frac{1}{r}\phi'(r)$$

to define the class of functions we shall investigate in the next sections. We start with the truncated power function $\phi_l(r) = (1-r)_+^l$ and then define

$$\phi_{d,k} = I^k \phi_{\lfloor d/2 \rfloor + k + 1} \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. In [17] the following theorem is proved and recursion formulae are given.

THEOREM 1.2. *The functions $\phi_{d,k}$ induce positive definite functions on $\mathbb{R}^d$ of the form*

$$\phi_{d,k}(r) = \begin{cases} p_{d,k}(r), & 0 \leqslant r \leqslant 1 \\ 0, & r > 1 \end{cases}$$

with a univariate polynomial $p_{d,k}$ of degree $\lfloor d/2 \rfloor + 3k + 1$. They possess continuous derivatives up to order $2k$. They are of minimal degree for given space dimension d and smoothness $2k$ and are up to a constant factor uniquely determined by this setting.

Thus these functions are the natural candidates for interpolation by compactly supported radial basis functions and a further investigation of their properties is necessary.

Though in [17] one kind of recursion formula is given to compute $\phi_{d,k}$ we want to add here a simpler formula, which can be proved by induction.

THEOREM 1.3. *Within its support $[0, 1]$ the function $\phi_{d,k}$ has the representation*

$$p_{d,k}(r) = \sum_{j=0}^{l+2k} d_{j,k}^{(l)} r^j$$

with $l = \lfloor d/2 \rfloor + k + 1$. The coefficients can be computed recursively for $0 \leqslant s \leqslant k-1$:

$$d_{j,0}^{(l)} = (-1)^j \binom{l}{j}, \qquad 0 \leqslant j \leqslant l$$

$$d_{0,s+1}^{(l)} = \sum_{j=0}^{l+2s} \frac{d_{j,s}^{(l)}}{j+2}, \qquad d_{1,s+1}^{(l)} = 0, \qquad s \geqslant 0$$

$$d_{j,s+1}^{(l)} = -\frac{d_{j-2,s}}{j}, \qquad s \geqslant 0, \quad 2 \leqslant j \leqslant l+2s+2.$$

Furthermore, precisely the first k odd coefficients $d_{j,k}^{(l)}$ vanish.

For convenience, we list the simplest cases in Table I, where $\doteq$ denotes equality up to a positive constant factor.

TABLE I

Table of Functions

$d=1$	$\phi_{1,0}(r)=(1-r)_+$	C^0
	$\phi_{1,1}(r)\doteq(1-r)_+^3\ (3r+1)$	C^2
	$\phi_{1,2}(r)\doteq(1-r)_+^5(8r^2+5r+1)$	C^4
$d=3$	$\phi_{3,0}(r)=(1-r)_+^2$	C^0
	$\phi_{3,1}(r)\doteq(1-r)_+^4\ (4r+1)$	C^2
	$\phi_{3,2}(r)\doteq(1-r)_+^6\ (35r^2+18r+3)$	C^4
	$\phi_{3,3}(r)\doteq(1-r)_+^8\ (32r^3+25r^2+8r+1)$	C^6
$d=5$	$\phi_{5,0}(r)=(1-r)_+^3$	C^0
	$\phi_{5,1}(r)\doteq(1-r)_+^5\ (5r+1)$	C^2
	$\phi_{5,2}(r)\doteq(1-r)_+^7\ (16r^2+7r+1)$	C^4

2. ERROR ESTIMATES

Knowing that interpolation is always possible, it is necessary to look for the behaviour of the interpolation error $f-s_f$ (pointwise or in a given norm) as a function of the data density. Therefore the space of functions f to be approximated (or interpolated) has to be specified and a "measure" for the data density has to be introduced. Naturally, the function space will depend on the basis function Φ and we will denote it by $\mathscr{R}_\Phi$. The density-"measure" for a set of centers $X=\{x_1, ..., x_N\}\subseteq\Omega\subseteq\mathbb{R}^d$ will be of the form

$$h=\sup_{x\in\Omega}\ \min_{1\leqslant i\leqslant N}\ \|x-x_i\|_2 \tag{4}$$

if we concentrate on a compact subset Ω of $\mathbb{R}^d$ satisfying a uniform interior cone condition.

There are several papers studying this kind of approximation problem by introducing the right space, often called "native" space, and then giving approximation orders depending on h. We cite for example [4, 8, 9, 11, 19].

Here, we follow [19] because it serves our purposes best and it will come out that the native spaces for our functions are norm-equivalent to Sobolev spaces (see Theorem 2.1). We start with a positive definite and integrable function Φ and define its Fourier transform by

$$\hat{\Phi}(\omega)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\Phi(x)\,e^{-ix^T\omega}\,dx.$$

Then the native space $\mathscr{R}_\Phi$ consists of all generalized functions $f\colon\mathbb{R}^d\to\mathbb{R}$ which can be recovered via

$$f(x)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\hat{f}(\omega)\,e^{ix^T\omega}\,d\omega,$$

where $\hat{f}$ is a function satisfying

$$\hat{f}/\sqrt{\hat{\Phi}} \in L_2(\mathbb{R}^d).$$

The norm on $\mathcal{N}_\Phi$ is given by

$$\|f\|_\Phi^2 := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2}{\hat{\Phi}(\omega)}\, d\omega.$$

Equipped with this norm, which obviously comes from an inner product, $\mathcal{N}_\Phi$ becomes a Hilbert space. From [19] it follows that if $\hat{\Phi}$ has an asymptotic behaviour like

$$\hat{\Phi}(\omega) = \mathcal{O}(\|\omega\|_2^{-d-s_\infty}), \qquad \|\omega\|_2 \to \infty, \tag{5}$$

then for every $f \in \mathcal{N}_\Phi$ and every sufficiently small $h \leqslant h_0$ the error estimate

$$\|f - s_f\|_{L_\infty(\Omega)} \leqslant Ch^{s_\infty/2} \|f\|_\Phi \tag{6}$$

is valid.

While upper bounds like (5) on the Fourier transform yield bounds for the approximation order, the corresponding lower bounds for the Fourier transform imply upper bounds for the stability of the interpolation process. Thus it is of enormous importance for the native space, the numerical stability, and the approximation order to know the precise asymptotic behaviour of the Fourier transform of the underlying radial basis function.

Now, let us assume we have already proved the following relations for the compactly supported radial functions $\Phi_{d,k}(x) = \phi_{d,k}(\|x\|_2)$, $x \in \mathbb{R}^d$, of the last section

$$\hat{\Phi}_{d,k}(\omega) \leqslant C_1 \|\omega\|_2^{-d-2k-1}, \qquad \|\omega\|_2 > 0, \tag{7}$$

and

$$\hat{\Phi}_{d,k}(\omega) \geqslant C_2 \|\omega\|_2^{-d-2k-1}, \qquad \|\omega\|_2 > r_0 \tag{8}$$

with certain constants $0 < C_2 \leqslant C_1$ and r_0. Actually, relation (8) does not hold in the case $k = 0$ if the space dimension is $d = 1$ or $d = 2$, but this is no serious restriction. If we also know that the Fourier transform $\hat{\Phi}_{d,k}$ is always positive we can find other constants K_1 and K_2 with

$$K_1(1 + \|\omega\|_2^2)^{-d/2-k-1/2} \leqslant \hat{\Phi}_{d,k}(\omega) \leqslant K_2(1 + \|\omega\|_2^2)^{-d/2-k-1/2} \tag{9}$$

for all $\omega \in \mathbb{R}^d$ and this yields

THEOREM 2.1. *Let $\phi_{d,k}$ denote the compactly supported radial basis function in $\mathrm{PD}_d \cap C^{2k}(\mathbb{R})$ of minimal degree and let $k \geqslant 1$ for $d = 1, 2$. The native space $\mathscr{R}_{\phi_{d,k}}$ belonging to $\phi_{d,k}$ then coincides with Sobolev space $H^s(\mathbb{R}^d)$ with $s = d/2 + k + 1/2$ and the native space norm is equivalent to the Sobolev norm. The Fourier transform satisfies* (9).

We remark in passing that the usual Sobolev embedding theorem for this case yields

$$\mathscr{R}_{\phi_{d,k}} \subseteq C^k(\mathbb{R}^d).$$

The approximation order also results immediately from (6) and from the relations (7) and (8). We state it only if $k \geqslant 1$ for $d = 1, 2$, even if the native space version is also valid in these cases.

THEOREM 2.2. *Let $s = d/2 + k + 1/2$ and $k \geqslant 1$ for $d = 1, 2$. For every $f \in H^s(\mathbb{R}^d)$ and every compact $\Omega \subseteq \mathbb{R}^d$ satisfying a uniform interior cone condition the interpolant $s_{f,X}$ on $X = \{x_1, ..., x_N\} \subseteq \Omega$ satisfies the estimate*

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \leqslant C \|f\|_{H^s(\mathbb{R}^d)} h^{k+1/2} \tag{10}$$

with h defined as in (4) *sufficiently small. Thus interpolation with $\phi_{d,k}$ provides at least approximation order $k + 1/2$.*

Note that this approximation order is comparable to that of thin-plate-spline or polyharmonic spline interpolation if a comparably smooth basis function is chosen.

As a simple example we apply Theorem 2.2 to the C^2-function $\phi_{3,1}(r) = (4r+1)(r-1)_+^4$ in $\mathbb{R}^3$ and get approximation order 3/2 for functions from $H^3(\mathbb{R}^3)$ which equals the approximation order of interpolation with cubics $\phi(r) = r^3$.

Before we prove the estimates (7) and (8) in the next section we state one more consequence. We already know that $\phi_{d,k}$ is in $C^{2k}(\mathbb{R})$, but now (7) and (8) lead also to

COROLLARY 2.3. *The induced functions $\Phi_{d,k}(x) = \phi_{d,k}(\|x\|_2)$, $x \in \mathbb{R}^d$, are in $C^{2k}(\mathbb{R}^d)$.*

3. ASYMPTOTIC BEHAVIOUR OF THE FOURIER TRANSFORM

The right tool to handle compactly supported functions in this context is the Fourier transform. A famous theorem of Bochner shows that positive definite and integrable functions are characterized by a nonnegative,

nonvanishing Fourier transform. We introduce the operator $\mathcal{F}_d$, which actually operates on univariate functions by

$$(\mathcal{F}_d\phi)(r) \equiv \hat{\Phi}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(\omega)\, e^{-ix^T\omega}\, d\omega$$

$$= r^{-(d-2)/2} \int_0^\infty \phi(t)\, t^{d/2} J_{(d-2)/2}(rt)\, dt$$

with $\Phi(x) = \phi(r)$, $r = \|x\|_2$, and the Bessel function of the first kind

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\, \Gamma(\nu+m+1)}.$$

Thus $\mathcal{F}_d\phi$ is the radial function representing the d-variate Fourier transform of the radial function $\Phi(x) = \phi(\|x\|_2)$.

It was first observed in [18] that the operators I and D of the last section influence the Fourier transform in the following way,

$$\mathcal{F}_d I = \mathcal{F}_{d+2}, \qquad \mathcal{F}_d D = \mathcal{F}_{d-2}, \tag{11}$$

whenever the operations are defined. So the d-variate Fourier transform of a function $\mathbb{R}^d \ni x \mapsto \phi(\|x\|_2)$ coincides with the $(d-2)$-variate Fourier transform of the function $\mathbb{R}^{d-2} \ni x \mapsto (I\phi)(\|x\|_2)$ and a similar relation holds for ϕ and $D\phi$. This was used to construct the functions of minimal degree in [17] and will now be used to investigate their Fourier transforms $\mathcal{F}_d\phi_{d,k}$.

We start by setting $l = \lfloor d/2 \rfloor + k + 1$ and get

$$\mathcal{F}_d\phi_{d,k}(r) = \mathcal{F}_d I^k \phi_l(r) = \mathcal{F}_{d+2k}\phi_l(r)$$

$$= r^{-d-2k-l} \int_0^r (r-t)^l\, t^{d/2+k} J_{d/2+k-1}(t)\, dt.$$

Now we have to distinguish between odd and even space dimension d. We first investigate the odd dimensional case. Therefore we set $d = 2n+1$, $m = k+n$, and assume m to be greater than zero, which excludes only the C^0-function in $\mathbb{R}$. On account of $l = m+1$ we get

$$\mathcal{F}_{2n+1}\phi_{d,k}(r) = r^{-3m-2} \int_0^r (r-t)^{m+1}\, t^{m+1/2} J_{m-1/2}(t)\, dt$$

$$= \mathcal{F}_{2m+1}\phi_{m+1}(r). \tag{12}$$

We first turn to lower bounds for this Fourier transform.

LEMMA 3.1. *For $m>0$ there exists a constant $C(m)$ such that for $r>0$ the lower bound*

$$\mathcal{F}_{2m+1}\phi_{m+1}(r) \geqslant C(m)\, r^{-2m-1}[J^2_{m+1/2}(r/2)+J^2_{m+3/2}(r/2)]$$

is valid.

Proof. We use formula (2.7) in [7] with $\alpha = m+1/2$ to get

$$\begin{aligned}
&\int_0^r (r-t)^{m+1}\, t^{m+1/2} J_{m-1/2}(t)\, dt \\
&\quad = (m+1)\int_0^r (r-t)^m\, t^{m+1/2} J_{m+1/2}(t)\, dt \\
&\quad = (m+1)\,\frac{\Gamma(m+1)\,\Gamma(2m+2)\,\Gamma(m+3/2)}{\Gamma(3m+3)}\, 2^{3m+3/2} r^{m+1} \\
&\qquad \times \sum_{j=0}^{\infty} \frac{((2m+2)/4)_j\,(2m/4)_j}{((6m+6)/4)_j\,((6m+8)/4)_j}\,\frac{(2m+2)_j}{j!}\,\frac{2j+2m+1}{j+2m+1}\, J^2_{m+j+1/2}(r/2)
\end{aligned}$$

after one integration by parts. As each factor on the right-hand side is positive we can neglect all but the first two terms of the sum to gain a lower bound of the stated form. ∎

Since the zeros of different Bessel functions do not coincide the bound is strictly positive. The next step for lower bounds is to bound the sum of Bessel functions from below.

LEMMA 3.2. *For $m \in \mathbb{N}_0$ there exists an r_m such that for all $r \geqslant r_m$*

$$J^2_{m+1/2}(r)+J^2_{m+3/2}(r) \geqslant \frac{1}{\pi r}$$

holds.

Proof. As the index of the involved Bessel functions is half an odd integer, the Bessel functions can be represented by a finite sum of the form

$$J_{m+1/2}(r) = \left(\frac{2}{\pi r}\right)^{1/2} \sum_{j=0}^{m} \alpha_{j,m} f_{j,m}(r)\, r^{-j}, \tag{13}$$

where $\alpha_{j,m} = (-1)^{\lfloor d/2 \rfloor}\,(m+j)!/(j!(m-j)!\,2^j)$ for $0 \leqslant j \leqslant m$ and $f_{j,m}$ is alternatively the function $\sin(\cdot - \pi m/2)$ or $\cos(\cdot - \pi m/2)$ if j is even or odd,

respectively (cf. [16]). We assume $\alpha_{j,m}$ to be zero in all other cases for convenience. A simple squaring yields

$$J_{m+1/2}^2(r) = \frac{2}{\pi r}\left(f_{0,m}^2(r) + \sum_{j=1}^{2m} \beta_{j,m}(r)\, r^{-j}\right)$$

with uniform bounded coefficients $\beta_{j,m}(r)$. If we now add two neighbouring squares we gain

$$J_{m+1/2}^2(r) + J_{m+3/2}^2(r) = \frac{2}{\pi r}\left(f_{0,m}^2(r) + f_{0,m+1}^2(r) + \sum_{j=1}^{2m+2} \gamma_{j,m}(r)\, r^{-j}\right),$$

where the coefficients $\gamma_{j,m}(r)$ are uniformly bounded again. If we finally take into account that $f_{0,m}^2(r) + f_{0,m+1}^2(r) = \sin^2(r - \pi m/2) + \cos^2(r - \pi m/2) = 1$ we only have to choose r_m large enough to get the stated inequality. ∎

Note that we have given a lower bound in Lemma 3.2, whereas an upper bound of this form is not surprising, since it is well known that each square is bounded by C/r. We also want to make the reader aware of the fact that formula (13) allows an explicit representation of $\mathcal{F}_d\phi_{d,k}$ for odd space dimension d, but we skip the details, since we are mainly interested in the general behaviour.

We will summarize the previous results in

PROPOSITION 3.3. *For odd space dimension $d = 2n+1$ and given smoothness $2k$ with $n + k \geq 1$ there exist constants $c_1 = c_1(d,k)$ and $r_0 = r_0(d,k)$ such that for all $r \geq r_0$ the inequality*

$$\mathcal{F}_d\phi_{d,k}(r) \geq c_1 r^{-d-2k-1}$$

is valid.

We now turn to upper bounds still in the odd dimensional case. We set $f_0(r) = 1 - \cos(r)$ and for $m > 0$

$$f_m(r) = f_0 * f_{m-1}(r) = \int_0^r f_0(t)\, f_{m-1}(r-t)\, dt. \tag{14}$$

By calculating the Laplace transform $\mathcal{L}$ of f_m and of the integral appearing in $\mathcal{F}_d\phi_{d,k}$ with $d = 2n+1$ and $m = n+k$ we get (cf. [1])

$$(\mathcal{L}f_m)(r) = \frac{1}{r^{m+1}(1+r^2)^{m+1}}$$

and

$$\mathscr{L}\left(\int_0^s (s-t)^{m+1}\, t^{m+1/2} J_{m-1/2}(t)\, dt\right)(r) = \frac{B_m}{r^{m+1}(1+r^2)^{m+1}}$$

with

$$B_m = \frac{m!(m+1)!\, 2^{m+1/2}}{\sqrt{\pi}}.$$

Note, that both functions are of convolution type which simplifies the calculation of their Laplace transform. Finally we get

$$\int_0^r (r-t)^{m+1}\, t^{m+1/2} J_{m-1/2}(t)\, dt = B_m f_m(r). \tag{15}$$

Thus we have done the main work for our next proposition.

PROPOSITION 3.4. *The Fourier transform of $\phi_{d,k}$ for odd $d = 2n+1$ possesses the upper bound*

$$\mathscr{F}_d \phi_{d,k}(r) \leqslant c_2 r^{-d-2k-1}$$

for all $r > 0$ with a constant c_2 depending only on d and k.

Proof. This follows easily by induction from the equality $\mathscr{F}_d \phi_{d,k}(r) = B_m r^{-3m-2} f_m(r)$ with $m = k+n$, which is a consequence of (15). ∎

Thus the asymptotic behaviour of the Fourier transform of the compactly supported radial basis functions of minimal degree is completely known in case of odd dimensional spaces and we have to turn to even dimensional cases. We use the results for odd dimension in a similar way as we have gained the upper bounds there. Now we set $d = 2n$ and, again, $m = k+n$; then we have $m \geqslant 1$ because n is at least 1. The d-variate Fourier transform of $\phi_{d,k}$ now has the form

$$\mathscr{F}_d \phi_{d,k}(r) = r^{-3m-1} \int_0^r (r-t)^{m+1}\, t^m J_{m-1}(t)\, dt. \tag{16}$$

We again follow [1] to introduce

$$g_0(r) = \int_0^r J_0(t)\, dt$$

and $g_m(r) = g_0 * f_{m-1}(r)$, where f_m is the function defined in (14). We state some facts about the g_m's.

LEMMA 3.5. (1) *For $m = k + n$ we have $\mathcal{F}_{2n}\phi_{2n,k}(r) = B_{m-1} r^{-3m-1} g_m(r)$.*

(2) *$g_0(t) > 0$ for $t > 0$.*

(3) *There exists a t_0 such that $1/2 \leqslant g_0(t) \leqslant 3/2$ holds for all $t \geqslant t_0$.*

Proof. The first assertion can again be proved by comparing the Laplace transforms of g_m and of the integral in (16) (cf. [1]). The second assertion is an inequality originating from Cooke [3], whereas the last assertion follows from $\int_0^\infty J_0(t)\, dt = 1$ (cf. [6]). ∎

We are now able to prove the main result of this section.

THEOREM 3.6. *Let $\phi_{d,k}$ denote the compactly supported radial basis functions of minimal degree, positive definite on $\mathbb{R}^d$ and in C^{2k}. For every space dimension d and every $k \in \mathbb{N}_0$ there exists a constant c_2 depending only on d and k such that for all $r \geqslant 0$ the Fourier transform of $\phi_{d,k}$ satisfies*

$$\mathcal{F}_d\phi_{d,k}(r) \leqslant c_2 r^{-d-2k-1}.$$

If $d \geqslant 3$ for $k = 0$ or arbitrary otherwise then there exist constants c_1 and r_0 depending on d and k such that for $r \geqslant r_0$ the lower bound

$$c_1 r^{-d-2k-1} \leqslant \mathcal{F}_d\phi_{d,k}(r)$$

is also valid.

Proof. The odd dimensional case is treated in Propositions 3.3 and 3.4, so we only have to consider the even dimensional case $d = 2n$. Let C denote a generic constant and $m = n + k$. From Lemma 3.5 we get $g_0(t) \leqslant C$ for all $t \geqslant 0$. This yields with Proposition 3.4

$$g_m(r) = \int_0^r f_{m-1}(t)\, g_0(r-t)\, dt \leqslant C \int_0^r t^{m-1}\, dt = Cr^m$$

and this is valid for all $r > 0$ and all $m \in \mathbb{N}_0$ which includes the case $k = 0$ even for $d = 2$. We now turn to the lower bounds and assume $m > 1$. As f_{m-1} and g_0 are positive and $f_{m-1}(r) \geqslant Cr^{m-1}$ for $r \geqslant r_0$ and $g_0(r) \geqslant 1/2$ for $r \geqslant r_0$ we have

$$g_m(r) \geqslant \int_{r_1}^{r_2} f_{m-1}(t)\, g_0(r-t)\, dt \geqslant Cr^m,$$

where $0 \leqslant r_1 \leqslant r/2 \leqslant r_2 \leqslant r$ have to be chosen properly such that $r_1 \geqslant r_0$ and $r - r_2 \geqslant r_0$ which completes the proof. ∎

4. CONDITION NUMBERS

In this final section we shall investigate the behaviour of the condition number of the interpolation process. We can make use of the results of the previous section if we consider the norm of the inverse of the interpolation matrix. The results concerning the norm of the interpolation matrix are of a more general type and can be applied to all kinds of positive definite and compactly supported basis functions.

Knowing the lower bound (8) standard techniques dating back to Narcowich and Ward [10] (cf. also [12]) yield

THEOREM 4.1. *Let $\phi_l(r) = (1-r)_+^l$ and let $\phi_{d,k}(r) = I^k \phi_l(r) \in \mathrm{PD}_d \cap C^{2k}$ be the function of minimal degree with $l = \lfloor d/2 \rfloor + k + 1 > 1$. Let $X = \{x_1, ..., x_N\} \subseteq \mathbb{R}^d$ be a set of centers with separation distance $2q_X = \min_{i \neq j} \|x_i - x_j\|_2 > 0$ and denote the interpolation matrix with $A_{X,\phi_{d,k}} = (\phi_{d,k}(\|x_i - x_j\|_2))_{i,j}$. The norm of the inverse of the interpolation matrix then satisfies*

$$\|A_{X,\phi_{d,k}}^{-1}\|_2 = \mathcal{O}(q_X^{-2k-1}), \qquad q_X \to 0. \tag{17}$$

Thus the norm of the inverse of the interpolation matrix grows only polynomially in terms of the separation distance, if the latter tends to zero.

We now turn to the norm of the interpolation matrix itself. Here, we deal with a more general setting. Given a continuous function $\Phi: \mathbb{R}^d \to \mathbb{R}$ with support in the unit ball $B_1(0) := \{x \in \mathbb{R}^d : \|x\|_2 \leqslant 1\}$ and a set of scattered and pairwise distinct centers $X = \{x_1, ..., x_N\}$ we are interested in the norm of the matrix $A_{X,\Phi} = (\Phi(x_j - x_k))$. Without loss of generality we restrict ourselves to sets of centers X with separation distance $q_X < 1/2$, otherwise the matrix $A_{X,\Phi}$ is of diagonal form. To motivate our next theorem let us assume that Φ has infinity norm 1. Let us further assume that the centers are given on a regular grid with width $2q_X = 1/(N^{1/d} - 1)$ and that there are at least two centers in each direction of the grid. Then the Gerschgorin theorem yields immediately

$$\|A_{X,\Phi}\|_2 \leqslant N \leqslant q_X^{-d}.$$

A general theorem has to cover this asymptotic behaviour.

LEMMA 4.2. *If $\Phi: \mathbb{R}^d \to \mathbb{R}$ is a continuous function with support $B_1(0)$ and $\max |\Phi(x)| \leq 1$, if $X = \{x_1, ..., x_N\} \subseteq \mathbb{R}^d$ is further a set with pairwise distinct points and $q_X < 1/2$, then for all $x \in \mathbb{R}^d$*

$$\sum_{j=1}^{N} |\Phi(x - x_j)| \leq 4^d q_X^{-d}. \tag{18}$$

Proof. The proof uses a counting argument of Narcowich and Ward [10]. More precisely we set for fixed $x \in \mathbb{R}^d$

$$S_k(x) = \{x_j \in X: q_X k \leq \|x - x_j\|_2 < q_X(k+1)\}.$$

Then we have according to [10]

- $\text{card}[S_0(x)] \leq 1$,
- $\text{card}[S_k(x)] \leq 3^d k^{d-1}$ for $k \geq 1$.

If $x_j \in S_k(x)$ for a k with $kq_X \geq 1$, so $\Phi(x - x_j) = 0$. Thus we have

$$|\Phi(x - x_j)| \leq \begin{cases} 1, & k < 1/q_X \\ 0, & k \geq 1/q_X. \end{cases}$$

As X is covered by the union of $S_k(x)$ we derive

$$\begin{aligned} \sum_{j=1}^{N} |\Phi(x - x_j)| &\leq \sum_{k=0}^{\infty} \text{card}[S_k(x)]\, \chi_{[0, \lfloor 1/q_X \rfloor]}(k) \\ &\leq 1 + \sum_{k=1}^{\lfloor 1/q_X \rfloor} 3^d k^{d-1} \\ &\leq 1 + 3^d \lfloor 1/q_X \rfloor^d \leq 4^d (1/q_X)^d, \end{aligned}$$

which completes the proof. ∎

Of course this estimate is rather inaccurate, but it is sufficient for our purposes.

THEOREM 4.3. *Let $\Phi: \mathbb{R}^d \to \mathbb{R}$ be positive definite. Further let X, q_X, and $A = A_{X,\Phi}$ be defined as before. Then*

$$\|A_{X,\Phi}\|_2 \leq \left(\frac{4}{q_X}\right)^d$$

is valid.

Proof. If we use $a^2+b^2 \geqslant 2ab$ and the previous lemma we get

$$\begin{aligned}
\|A\|_2 &= \sup |\alpha^T A \alpha| \\
&\leqslant \sum_{j=1}^{N} \sum_{k=1}^{N} |\alpha_j|\,|\alpha_k|\,|\Phi(x_j - x_k)| \\
&\leqslant \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} |\Phi(x_j - x_k)|\,(|\alpha_j|^2 + |\alpha_k|^2) \\
&= \sum_{j=1}^{N} |\alpha_j|^2 \sum_{k=1}^{N} |\Phi(x_j - x_k)| \\
&\leqslant \left(\frac{4}{q_X}\right)^d.
\end{aligned}$$

The sup has to be taken over all $\alpha \in \mathbb{R}^N$ with $\|\alpha\|_2 = 1$. ∎

Thus the condition number of the interpolation matrix using the piecewise polynomial, positive definite, and compactly supported radial basis functions of minimal degree can be bounded by

$$\operatorname{cond}_2(A_{X,\phi_{d,k}}) \leqslant C q_X^{-d-2k-1}.$$

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